M Math Analysis of several variables Final examination 20-11-2017.

Answer all the 10 questions. Each question is worth 6 points.

If you are using any result proved in the class, you need to state it correctly. If the answer is an immediate consequence of a result quoted, then that result also needs to be proved.

1. Let $f, g : R^n \to R$ be differentiable functions. Define $F : R^n \to R^2$ by $F((x_1, ..., x_n)) = (f(x_1, x_2, 0, ..., 0), g(0, 0, x_3, x_4, ..., x_n))$. Use chain rule to show that F is differentiable.
2. Let $f : R^2 \to R$ be defined by $f(x, y) = \frac{x^2y^2}{x^4+y^2}$ if $(x, y) \neq 0$ and $f(0, 0) = 0$. Show that f is differentiable at $(0, 0)$, but $\frac{\partial f}{\partial y}$ is not continuous at $(0, 0)$.
3. State and prove Taylor's theorem for a real-valued function f defined on the open unit ball $B(0, 1)$ in R^2 which has continuous partial derivatives of order ≤ 3.
4. Let $f : D = \{(x, y) \in R^2 : x + y < 1\} \to R$ be a convex function. Suppose f is differentiable at $(x_0, y_0) \in D$. Show that $f(x, y) - f(x_0, y_0) \geq \nabla f(x_0, y_0).(x - x_0, y - y_0)$ for all $(x, y) \in D$.
5. Let $f : R^n \to R^n$ be a continuously differentiable function. Suppose $f'(0)$ is invertible. Show that the image under f of the open set $f'^{-1}(B(f'(0), \frac{1}{2\|f'^{-1}(0)\|}))$, is an open set.
6. Let $Q^3 = \{(x, y, z) \in [0, 1] \times [0, 1] \times [0, 1] : x + y + z \leq 1\}$. Show that $\int_{Q^3} e^{x+2y+3z} = \frac{(e-1)^3}{6}$.
7. Let $f : [0, 1] \times [0, 1] \to R$ be a continuous function. Let $F(x, y) = \int_{[0,x]\times[0,y]} f(s, t)d(s, t)$. Show that $\frac{\partial F}{\partial x}$ exists.
8. Let $f : (0, 1) \to \{x \in R^n : x_1^2 + ... + x_n^2 = 1\}$ be a differentiable mapping. Show that the dot product $f.f' = 0$.
9. Let ω, λ be continuously differentiable 2-form and a 3-form respectively. Show with full details that $d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + w \wedge d\lambda$
10. Let $\Phi : [0, 1] \times [0, \pi] \times [0, 2\pi] \to R^3$ be defined by

$$\Phi(r, \theta, \phi) = (rsin\theta cos\phi, rsin\theta sin\phi, rcos\theta).$$

Show that $\int_\Phi dx \wedge dy \wedge dz = \frac{4\pi}{3}$.